INERTIA AS A THEOREM IN GALILEO'S DISCORSI

Júlio C. R. Vasconcelos

Abstract

Contrary to widely held beliefs, it could be said that there is no principle of inertia in Galileo's *Discorsi* (*Two New Sciences*) on the grounds that although one can find a conception of inertia in this masterpiece, this conception does not act in the *Discorsi* as a principle or as any kind of demonstrative tool, except inside a "*scholium*", in which Galileo argues, in mathematical terms and by two different "approaches", for the truth of the so-called "double-distance rule".

Galileo's conception of inertia appears only in the second of these "approaches"; the first of them, however, is the most important one as Galileo tries to relate the mentioned rule to the development of the second "new science", based upon just one principle, which is not a principle of inertia.

Given that Galileo himself, subsequently to that *scholium*, mentions twice the double-distance rule as obtained "*ex demonstratis*", it is probably safe to say that this rule works as a theorem of inertia in the *Discorsi*. It is this theorem, instead of any law of inertia, that is used for demonstration in Galileo's science of motion, especially in its projectiles theory. Last but not least, the theorem also provides a definition for instant speed in a fall and a measure for horizontal speed in a projectile motion.

1 Introduction

Although Galileo Galilei is universally considered to be a forerunner of the law of inertia, there has been intense debate on how much he contributed to the establishment of this law. Such debate is of paramount importance because, as Dijksterhuis [1986 pp. 347-8] reminds us, "the change in the conception of inertia...probably constitutes the most important element of all in the transition from ancient and medieval to classical science". Therefore, it would be fundamental for the history of science to dwell on "the whole vexed question of whether Galileo's understanding of inertia was complete or whether he only prepared the way for it to such an extent that his successors had little difficulty in reaching it" [Dijksterhuis, 1986 p. 347].

One cannot help but agree with Dijksterhuis when he emphasises the important role that the law of inertia plays in modern science. Yet, one can reject the approach to research that assumes only one possible conception of conservation of motion and that seeks to determine whether this conception is thorough or simply drafted in Galileo's works – or in any precursor of Newton's *Principia*.

Indeed, it is not beyond the bounds of possibility that Galileo developed a concept of conservation of motion that would not be a mere draft of Newton's first law. In this view, it is worth searching for a complete and original concept of inertia in Galileo's writings.

This paper aims to follow the line of investigation above, and it will be focusing on the *Discorsi e Dimostrazioni Matematiche intorno a due Nuove Scienze* (1638), where, as is well known, the mature Galileo's science of motion was published. Needless to say, this paper does not intend to cover the whole of Galileo's thought concerning conservation of motion.

2 Inertia is no a principle of demonstration in the discorsi

One of the possible statements for what might be called "Galileo's law of inertia" is to be found in the "third day" of the *Discorsi*, in the scholium of the twenty-third proposition of the theory of "naturally accelerated motion" (proposition which we shall abbreviate to III-23). It is with the following words that Galileo [1989 p.197] presents us with his conception of conservation of motion:

> It may ...be noted that whatever degree of speed is found in the moveable, this is by its nature indelibly impressed on it when external causes of acceleration or retardation are removed...[1]

This noteworthy conception, however, does not occur again in the *Discorsi*. Furthermore, neither this nor any other law of conservation of motion is necessary as an axiomatic pillar for Galileo's science of motion.

[1] In my doctorate thesis (Vasconcelos, J. C. R., 1998) I seek to show that these words from *Discorsi* should be understood as the statement of an original Galilean law of inertia.

Indeed, in the propositions of "*giornata terza*", except for the aforementioned scholium, no use of any conceptions of conservation of motion can be found. In the "fourth day", whenever it is necessary to assure of the uniform nature of the horizontal motion, there enters the following physical-mathematical rule:

> A moveable which falls from rest a certain vertical or inclined distance, if diverted horizontally after such a fall, traverses in uniform motion, in a time equal to that in which it fell, a distance which is twice the distance covered in the fall.[2]

Schematically, for the more general case of an initial fall on an inclined plane, we have:

```
                             /A
                            /
                           / (d,T)
                          /
C - - - - - (2d,T) - - - - - B
```

The main aim of this paper is to show that this rule, known to specialists as "the double-distance rule", may be understood as a theorem of inertia.

Reasons will be provided so that one can believe that the foundations of this theorem may dispense with a principle of inertia, as we shall see that Galileo extracts the theorem from one of the propositions sustained by the principle he wants to be unique in his science of motion [Galilei, G., 1989 p. 162]:

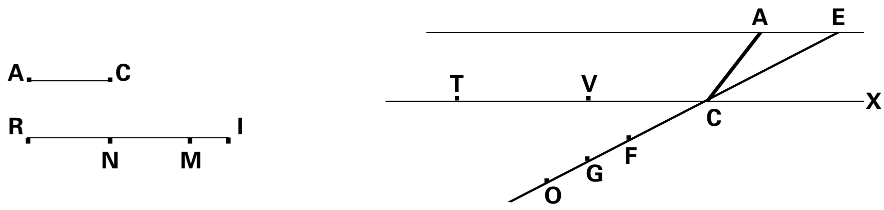
> ...the Author requires and takes as true one single assumption ("*principio*"); that is:
>
> I assume that the degrees of speed acquired by the same moveable over different inclinations of planes are equal whenever the heights of those planes are equal.

3 The first "approach" to demonstration of the theorem of inertia

It is in the scholium of proposition III-23 that Galileo brings to light the physical-mathematical formulation which is entitled "theorem of inertia" in this paper.

[2] Nowhere in the pages of the *Discorsi* is there to be found a formal statement for the rule: the above text was composed from several of Galileo's references to the regularity described by the rule.

In III-23 Galileo solves the problem of constructing an inclined plane of a given length IR which is covered after a fall AC, vertical or inclined,[3] in a time equal to that in which the fall took place. In the schema below, the two lines to the left are the problem data, given that RN = NM = AC > MI, which makes IR obey the condition of possibility 2AC < IR < 3AC, demonstrated in III-21. The figure on the right is the representation of the solution to the problem, a solution whose first step is to construct the line CE, with E on the same horizontal line as A, according to the proportion CE : AC :: MN : MI. On the extension of EC then, the solution CO is cut equal to RI, as requested.



In the proof that CO is the desired solution, one of the demonstrative steps is as follows [Galilei, G., 1989 p. 195]: "since it was assumed that the time through AC is as AC, the time through EC will be CE", a step which may only be justified by proposition III-3, the first proposition which is supported on the aforementioned principle (the previous ones are consequences of the definition of uniformly accelerated motion). Thus, this step is enough to assure that proposition III-23 is subordinate to the principle that Galileo wishes to be unique in his science of motion.

Moving on to the scholium of III-23: in the first paragraph of the scholium, Galileo argues [1989 p. 195-6] that when IR is given to be almost equal to 2.AC "then IM will be a very short line, ...AC will be very short with respect to CE, which will become very long, and nearly ...horizontal ..."[4]

After this reasoning, Galileo feels free to establish [1989 p. 196] his double-distance rule:

> And we may then deduce that if, in the above diagram, after descent through the inclined plane AC, there is diversion along a horizontal line such as CT, the space through which the moveable will next [*consequentur*] be moved, in

[3] In the statement and demonstration of proposition III-23, AC is said to be vertical but, immediately before the scholium, Galileo warns that "the same happens if the preceding motion is made not vertically, but along an inclined plane..." Note that in the quotations that follow, AC is sometimes vertical and sometimes inclined.

[4] In the opening paragraph, Galileo also remarks that "point E will be found close to point A" when IR is given to be almost equal to 3.AC (this is so because in this case MI is almost equal to MN and the aforementioned proportion –CE : AC :: MN : MI– makes the constructed line CE almost equal to AC).

> a time equal to that of descent through AC, would be exactly double the space AC.

To support this result, Galileo boldly seeks to demonstrate [1989 p. 196] that a finite proportion is still valid at the infinite limit:

> Further, it is seen that this fits with [other] like reasoning. For from the fact that OE is to EF as FE is to EC, it appears that FC determines the time through CO. For if the horizontal part TC, double CA, is bisected at V, its extension toward X will be prolonged indefinitely in seeking to meet with AE produced; and the ratio of an infinite TX to an infinite VX will not be different from the ratio of an infinite VX to an infinite XC.

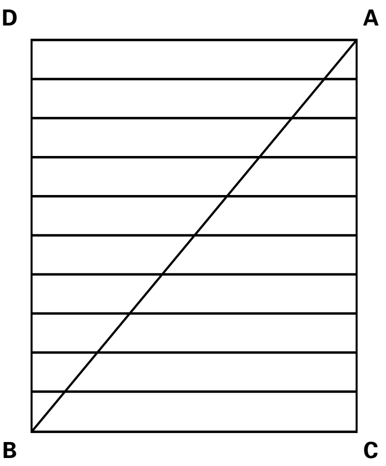
It was during one of the steps of the demonstration of III-23 that Galileo had arrived at the proportion stated above, OE : FE :: FE : EC. Galileo is now presenting the proportion in three infinite terms TX : VX :: VX : XC, where O seems to correspond to T, E to X and F to V, with X on the horizontal which passes through C, X being, however, infinitely separated from C. Nevertheless, these ratios between infinite lines have no mathematical foundations, as the Euclidean theory of proportions, in his fourth definition [Euclid, 1956, vol. II, pp. 113-20], forbids the comparison of infinitely large or infinitely small magnitudes.[5]

4 The second "approach" to demonstration

Let us move on, however, leaving aside for a moment Galileo's mathematical difficulties; we should note, then, that the first paragraph of the scholium failed to declare, at any moment, that motion along the horizontal plane is uniform. Indeed, by means of the construction that solves the problem proposed by III-23, there is no way of checking the uniformity of motion along the horizontal plane. However, the uniform nature of the horizontal motion is brought up in the following lines of the scholium, when Galileo decides [1989 p. 196] to demonstrate the theorem "by another approach, taking up again an argument like that which we used in the demonstration of Proposition I": [6]

[5] In addition to this daring proportion, one may also note an inversion of the terms of argumentation, as Galileo made, in the passage quoted above, the horizontality of the plane antecedent, whilst in the proposition the inclination of the plane is obtained and not given.

[6] *Istud idem* alia aggressione *concludere poterimus, consimile resumentes ratiocinium ei, quo usi sumus in propositionis primae demonstratione* (Galilei, G., 1933 vol. VIII p. 242).



For take again the triangle ABC, and by its parallels to the base BC let us represent to ourselves the degrees of speed continually increased according to the increments of time. From those, which are infinitely many...there arises the surface of the triangle [ABC]; and if we assume the motion to be continued for another equal time, no longer in accelerated but in equable motion at the maximum degree of speed acquired (which degree is represented by line BC), then from these degrees [of speed] a like parallelogram ABCD will be produced, double the triangle ABC. Hence the space which is traversed in the same time with similar degrees [of speed] will be double the space run through with the degrees of speed represented by triangle ABC.

What enables us, in this case, "[to] assume the motion to be continued...in equable motion"? This is Galileo's answer [1989 p. 196-7]:

> But motion in the horizontal plane is equable, as there is no cause of acceleration or retardation; therefore it is to be concluded that the space CT, run through in time equal to the time AC, is double the space AC...

A few lines later, Galileo reinforces [1989 p. 197] the above words with the concept of conservation of motion already mentioned at the beginning of this paper: "... whatever degree of speed is found in the moveable, this is by its nature indelibly impressed on it when external causes of acceleration or retardation are removed..."[7]

5 Why Galileo takes a risk at his stumbly first "approach"

The above concept, applied to horizontal motion, and the first proposition of accelerated motion (III-1), as has been seen, may be said to be sufficient as arguments to establish the double-distance rule as a 'theorem of inertia'. However, Galileo presents these arguments as being simply "another approach" to demonstration.

[7] ... "which occurs only on the horizontal plane", Galileo adds. This comment is generally taken to support the interpretation that Galileo's inertia is exclusively horizontal. Now what Galileo says here is that conservation of the degree of speed is manifest only on the horizontal plane (confusing the domain of application of a principle with its domain of explicit manifestation would only make Newtonian inertia, which as a rule never manifests itself (due to universal gravity and passive resistance), a principle with a non-existent spectrum of application).

In the *Dialogo* (1632), Galileo had already presented the double-distance rule, using figures and arguments almost identical to those of this second "approach" within the *Discorsi*. In that first masterpiece, this "approach" seems to be considered as being of secondary importance, as, upon ending his presentation, Galileo [1967, p. 229-230; 1933 vol. VII, p. 591-2] makes it clear that it is not "rigorous proof (*dimostrazione necessaria*)", but merely makes it "reasonable and probable (*ragionevole e probabile*)" that the distance travelled in the subsequent uniform motion is double the distance of the fall.

Why denigrate this "other approach" in the **Dialogo**, apparently so much more complete and secure, and why take the risk, in the *Discorsi*, of ratios between infinites, in the first paragraph of the scholium of III-23?

Perhaps because proposition III-1, used throughout the "other approach" of demonstration of III-23, is independent of the unique principle, as it is deduced directly from the definition of naturally accelerated motion. Thus, Galileo has no alternative apart from the first "approach" to associate the theorem of inertia to the unique physical principle of his science of motion.

And the proof of the physical validity of the theorem of inertia is necessary because:

a) it is one of the results of the "third day" which will be used in the mathematical development of the projectiles theory;
b) the remaining support for the projectiles theory –the theorems of uniform motion and the second corollary of III-2, the geometric equivalent of $s^{1/2} = K.t$ – do not depend on the unique principle.[8]
c) the unique principle must form the basis for the projectiles theory as this theory completes and is the apex of the second of the "*two new sciences*" of the *Discorsi*.

Thus, the theorem of inertia cannot fail to be associated with the principle postulated, if the latter is complete and sufficient as the physical pillar itself that sustains the new science of motion. Even at the cost of having to deal with ratios between non-finite lines, Galileo seeks arguments in order to demonstrate, or at least provide evidence for the adequacy of the theorem of inertia to the system of propositions which stems from the unique principle.[9]

[8] In my master's degree dissertation (Vasconcelos, J. C. R.,, 1992) between pages 107 and 166, I undertake an analysis of the steps of the demonstrations in the fourth *giornata*, seeking to establish the foundations of each one. It was at the end of this task that I came across this noteworthy characteristic of projectiles theory. In that dissertation I also argue that demonstration of the parabolic shape of projectiles trajectory in IV-1 uses no principle or concept of inertia, inasmuch as the uniformity of horizontal motion therein is given *ex suppositione*. At the same time, the rule of composition of motion demonstrated in IV-2 may be said to be strictly kinematic, as it is supported solely by the theory of uniform motion. Thus, the *only truly physical instrument employed in the demonstrations of projectiles theory is the theorem of inertia.*

[9] It is taken as read that he is unable, through this first "approach" to demonstration, to

6 Wisan and the double-distance rule as a first "foundation"

According to Winifred Wisan [1974, p. 277], the double-distance rule served as an original foundation for Galileo's science of motion:

> ... GALILEO appears to have been working over his treatise on motion as he was finishing the dialogue on astronomy. This is suggested by remarks in his correspondence...and by references throughout the dialogue... There is no hint [in these remarks and references] of Theorem I on accelerated motion and some last minute changes in this theorem suggest that it was revised and added to the treatise immediately before publication. It may be, then, that about 1630 GALILEO had a foundation for his treatise on motion which was based on the double-distance rule...

Therefore, the very law of fall was firstly deduced from this rule as "the published proof of the times-squared theorem was first based on the double-distance rule and only later altered slightly so as to follow from Theorem I instead" [Wisan 1974 p. 280].

As Wisan has pointed out, this replacement was done when the *Discorsi* were about to be printed, and she believes [1974 p. 220] that the sudden substitution of Theorem I for the double-distance rule was necessary because "in the published text GALILEO carefully defines uniform motion so that the rules for uniform motion cannot be applied to accelerated motion except through Theorem I".

In the present paper, we are focusing on axiomatic aspects of "the published text" that may provide other possible reasons for that replacement: Galileo probably realised that he could turn the double-distance rule into a mathematical consequence of his "*solo principio*" throughout the first "approach" to demonstration of the theorem of inertia in the scholium of III-23. Nevertheless, the double-distance rule continues being one of the two demonstrative tools that the theory of "naturally accelerated motion" offers to the "*giornata quarta*" of the *Discorsi*.

The rule is no longer a foundation for the other tool, the second corollary of III-2 (the times-squared theorem); yet, this loss of axiomatic elegance does not matter: the double-distance rule is now the *physical* support that was missing in the projectiles theory, since in "the published text" this rule is related to the "*principio*" by the first "approach" in the scholium of III-23.

---

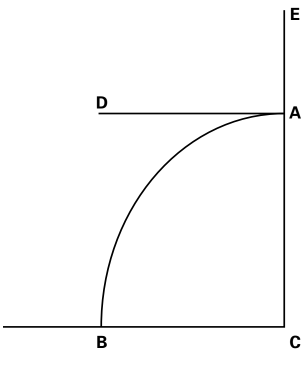
show the uniform character of horizontal motion. This first "approach" is, then, problematic and deficient, but is indispensable, for the reasons given above.

7 Uses of the theorem in the *Giornata* on projectiles

In the "third day" of the *Discorsi*, the theorem of inertia is employed in the second part of the scholium of III-23, in demonstrations of III-24 and III-25 and in III-29. But it is in the *giornata quarta* that this theorem has the most noble uses: it is an important part of the demonstration of propositions III, IV and V of projectiles theory (IV-3, IV-4 and IV-5), which, together with the first two (IV-1 and IV-2), form the set of fundamental propositions for this theory. These propositions may be said to be fundamental because, after the *quod quaerebatur* of IV-5, they become the main physical-mathematical sustainers of the theory, allowing references to propositions from the previous "day" to practically disappear.[10]

The theorem of inertia also plays another important role in the *giornata quarta*, revealed by Galileo [1989, pp. 231-2] in a remark prior to the demonstration of IV-4: the theorem enables Galileo to point out, from amongst the "inumerable degrees of speed for equable [horizontal] motions", exactly the one that should compound, along with the "naturally accelerated downward" motion, the parabolic trajectory:

> From that multitude [of degrees of speed for horizontal motion] I may select one and segregate it from the rest, as if pointing a finger at it, by extending upward the altitude CA, in which, whenever necessary, I shall fix the 'sublimity' AE.



> Now if I mentally conceive something falling from rest at E, it is evident that the impetus it acquires at terminus A is identical with that with which I conceive the same moveable to be carried when [it is] turned through the horizontal AD. This is that degree of swiftness with which, in the time of fall through EA, it would traverse double that distance EA in the horizontal. This prefatory remark I consider necessary.[11]

It is worth noting that here there is also an equivalent of a definition of instantaneous speed in a naturally accelerated motion. There is no other in the *Discorsi*, because:

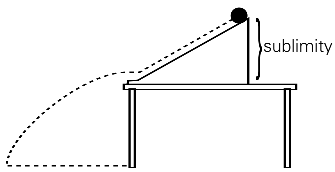
---

[10] This is one of the features of the *Discorsi* projectiles theory that I try to evidence in Vasconcelos, J. C. R., 1992, p. 120-166.

[11] It is also worthy of note that, following this remark, Sagredo delays the demonstration of IV-4 a little more, with an "adorn to the Author's thought", a cosmogonic myth which purports to be "a conception of Plato's". This position of myth in the *Discorsi* strengthens Eric Meyer's interpretation (Eric Meyer, "Galileo's Cosmogonical Calculations", *Isis*, 1989, 80: 456-468) that Galileo effectively sought to calculate the cosmogonic sublimity, one of the instruments of calculus being the double-distance rule, that is, the theorem of inertia.

a) Galileo is unable to define speeds as ratios between space and time, due to the prohibition of the Euclidean theory of proportions to ratios between magnitudes of different types;
b) Galileo can't draw the definition of instantaneous speed through the paradigms we know because in 1638 he was still a few decades from the foundation of Calculus by Newton and Leibniz.

It should be also noted that the theorem of inertia offers an advantage, associated with its first "approach" to demonstration, on the Kinematics we know: it does not correlate two abstract motions, but rather associates two motions with a physical meaning. The theorem may, therefore, be applied to the basic physical-experimental scheme of the *giornata* on projectiles,[12] in which the horizontal speed of launching is obtained by a fall through a vertical or inclined *"sublimity"*.



Thus one does not have here mere mathematical composition of two motions postulated independently; the horizontal motion of the projectile is derived from the natural vertical motion and should obey the laws dictated by this. The *giornata* on projectiles thus becomes the crowning of the theory of naturally accelerated motions, a unification which may be the most beautiful and original feature in Galileo's science of motion.

8 Conclusion

"Scholium", from the Greek "skólion", means "explanation" or "enlightenment"; however, in this paper, as may be seen, we are allowing ourselves to understand the "enlightenment" which follows III-23 as a theorem.

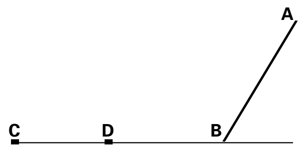
There are a number of justifications for this terminological daring: the first of these is that the result of that "enlightenment" has some characteristics of a theorem, although it comes, as has been shown, from a geometric problem. That is, although originating from a proposition intended basically to construct a figure, this result ends up being a more general property which, as seen in the above lines, is used in demonstrations as if it were more all-encompassing and more powerful than the construction technique taught in the proposition.[13]

[12] Which returns to a privileged position with the rediscovery in the seventies, by Stillman Drake, of important manuscripts by Galileo.

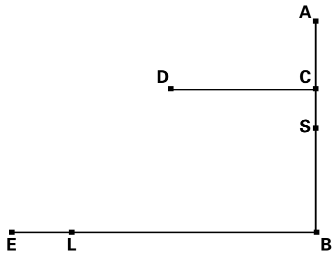
[13] The scholium of III-23 is not a corollary, as it is obtained through new arguments, whilst, according to the teachings of Heath, a corollary is "an accidental result which leaps from proof of a theorem or of the solution to a problem, a result not directly sought but which appears as if from luck without any additional work" (cf. note by T. Heath in Euclid , vol. I p. 278).

It is reasonable to understand that Galileo doesn't extract the "theorem of inertia" as a declared theorem, simply because in 1638 he did not have well based mathematical instruments for such a deduction.

There are at least two indicators that Galileo himself, in spite of the difficulties of arguments which are raised in the scholium of III-23, considers the "explanation" that one finds there as a demonstrated result. The first indicator lies [Galilei, G., 1989 p. 200] within proposition III-25 and the other appears [Galilei, G., 1989, p. 231] in an important passage of the demonstration from IV-3 (my underlining):



III-25 - "...For take BC double AB, and it follows from *what was demonstrated* above that the time of fall through AB equals the time of motion through BC; ..."[14]

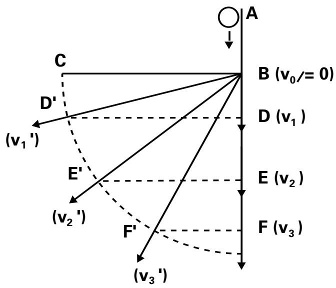


IV-3 – "...Draw the horizontal line CD (double AC) and BE (double BA); it follows from *what has been demonstrated* that a [moveable] falling through AC, turned into the horizontal CD and carried in equable motion according to the impetus acquired at C, traverses space CD in a time equal to that in which AC was traversed in accelerated motion. Similarly, BE is traversed in the same time as AB."[15]

If the reader, encouraged by Galileo's words above, ends up agreeing that the unique principle supports the demonstration of the theorem of inertia and that the latter is enough of a mathematical resource to deal with inertial motions, then one is able to accept for the old question "why didn't Galileo develop and announce the rectilinear principle of inertia in the *Discorsi*?", an answer of disconcerting simplicity: *he didn't do so because it was not necessary*, given that for him, indeed, the principle he said to be unique was enough.

[14] In the original Latin text (the language in which Galileo writes his theorems) the underlined passage corresponds to "*constat ex praedemonstratis*" (Galilei, G., 1933, vol VIII, p. 246). In the Latin-Portuguese Dictionary by F. Torrinha [*Dicionário Latino Português*, Gráficos Reunidos, Porto, 1989] one learns that the verb used by Galileo means "to be firm, to be firmly established, to be evident". In the French translation by Maurice Clavelin, we read "d'après une proposition déjà démontrée" [Galilée, G., 1970 p. 182].

[15] In Latin the underlined expression is "*constat, ex demonstratis,*" (Galilei, G., 1933, vol. VIII, p. 281). Clavelin translates it to "il est évident, par les démonstrations précédentes" (Galilée, G., 1970 p. 216)



To reinforce the perception of the sufficiency of Galileo's principle, let us consider the diagram to the right, in which a ball leaves point of rest A, falling vertically towards F or, diverting at B, following inclined planes BD', BE' or BF' in which initial speed (v_0) is not zero; as the regularity expressed through Galileo's unique principle is also valid for v_0 π 0 –this is a gain brought by proposition X (III-10) on his theory of naturally accelerated motion– we are therefore able to state that velocities v_1', v_2', v_3' are respectively equal to v_1, v_2, v_3.

When D is taken ever closer to B, v_1 tends to become equal to v_0; then, according to the principle, one may conclude that v_1', which is always equal to v_1, also tends to become equal to v_0. And thus, as D approaches C and the inclined plane BD' tends towards the horizontal BC, the motion tends towards uniformity.

The above argument, evidently, may be accused of being merely qualitative. However, in the body of this paper, mathematical reasons have been given in order to make it possible to understand why in the *Discorsi* "the Author requires and takes as true one single assumption" [Galilei, G., 1989 p.162]. These reasons may therefore enable one to revalue, as an instrument of exegetic interpretation, the confidence, in principle, in the declarations of scientific nature made by great men such as Galileo Galilei.

Bibliographical references

Dijksterhuis, E. J., *The Mechanization of the World Picture*, Princeton University Press, 1986.

Euclid, *The Thirteen Books of the Elements*, Dover, New York, 1956.

Galilée, G., *Discours e Démonstrations Mathématiques Concernant Deux Sciences Nouvelles*, Armand Colin, Paris, 1970.

Galilei, G, *Opere*, G. Barbèra, Firenze, 1933.

—, *Discorsi,* Ed. A cura de A. Carugo e L. Geymonat, Paolo Boringhieri, Torino, 1958.

—, *Dialogue Concerning the Two Chief Systems*, University of California Press, Berkeley, 1967.

—, *Two New Sciences*, 2d. ed., Toronto 1989.

Galileu, G., *Duas Novas Ciências*, Nova Stella, S. Paulo, 1988.

Newton, I., *Principia*, University of California Press, 1962.

Vasconcelos, J.C.R., *Uma Análise da teoria dos projéteis dos Discorsi de Galileo*, Masters' dissertation, FFLCH-USP, 1992.

—, *Inércia nos Discorsi e no Dialogo de Galileo*, Ph.D. dissertation, FFLCH-USP, 1998.

Wisan, W., "The New Science of Motion: A Study of Galileo's *De Motu Locali*", *Archive for History of Exact Sciences,* vol 13, pp. 103-306.